ISRAEL JOURNAL OF MATHEMATICS, Vol. 58, No. 1, 1987

MODULI OF COMMUTATIVE AND NON-COMMUTATIVE FINITE COVERS

BY

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ABSTRACT

We classify affine, not necessarily commutative, n-covers B of commutative K-algebras A using data triples (A, M, α) consisting of the algebra A, a free A-module M of rank $n-1$, and an associative, unitary trace-zero structure constant tensor α. We construct a versal deformation space for the deformations of a K-algebra B_0 as a section of the completion at the tensor α_0 of B_0 of the structure-constant scheme C_n. In order to obtain concrete information about the algebraic structure of C_n, we show how this algorithm has been implemented up to order 2. Finally, we globalize and geometricize the construction, getting a one-to-one correspondence between isomorphism classes of global n-covers and isomorphism classes of triples $(\mathcal{O}_Y, \mathcal{E}, \alpha)$, where $\mathcal{O}_Y$ is the structure sheaf of a commutative integral scheme Y, $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_Y$-modules of rank $n-1$, and α is a global section of a sheaf $\mathcal{H}^0_{n-1}(\mathcal{E})$ of structure constant tensors. We give examples in dimensions $n = 2, 3$, and 4 to show how the structure of $\mathcal{H}^0_{n-1}(\mathcal{E})$ can be analyzed as a functor of $\mathcal{E}$ using information about C_n obtained as above.

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§0. Introduction

This paper continues a very traditional line of research, the classification of n-dimensional algebras over an algebraically closed field K, but it is viewed in

Received December 1, 1985 and in revised form July 16, 1986

the light of more modern developments, the Schlessinger deformation theory and the non-commutative algebraic geometry of Artin and Schelter [1].

Our basic object of study is what we call an affine n-cover, an algebra B over a commutative K-algebra A, which is a free A-module of rank n. Our first main result is a generalization for all n of the "double cover data" in classical algebraic geometry. Given an A-module M of rank $n-1$, we introduce a functor $\mathcal{K}^0_{n-1}(M)_A$ which consists of all structure constant tensors α which put a multiplication on $e_0A \oplus M$ in such a way that M is the canonical submodule of trace zero elements under left multiplication. The affine version of the theorem then states

THEOREM 1′. *The isomorphism classes of affine n-covers are in one-to-one correspondence with the isomorphism classes of data triples (A, M, α).*

In the final sections of the paper we replace the ring A by a sheaf of rings over a (commutative) integral scheme Y of finite type, and we replace the free A-module M by a locally free sheaf $\mathscr{E}$ on Y of rank $n-1$. After sheafifying the functor $\mathcal{K}^0_{n-1}$ and letting α be a global section of $\mathcal{K}^0_{n-1}(\mathscr{E})$, we then get our global version of the result.

THEOREM 3. *The isomorphism classes of global n-covers are in one-to-one correspondence with the isomorphism classes of data triples $(\mathcal{O}_Y, \mathscr{E}, \alpha)$.*

This theorem is only interesting for general n insofar as we can obtain information about $\mathcal{K}^0_{n-1}(\mathscr{E})$, and this is actually just a restatement of the classical problem of classifying the irreducible components of the structure constant variety C_n. Thus in the middle section of the paper we apply the Schlessinger deformation theory to give a parameter space for the deformations of an algebra. This construction has been implemented on a microcomputer out to second order terms, and we work out one example in detail to show that the calculation is not only feasible but actually worth doing. The author has already used the program in various contexts, usually for generating data on which to conjecture general results about the classification and deformation of n-dimensional algebras.

As final applications, we give globalized parametrizations of the semi-rigid family in dimension 4 and the Kronecker component in all dimensions. We also give a geometric interpretation of the global n-covers in Theorem 3, using the maximal spectrum as the underlying geometric object.

The material is divided into sections as follows: §1 contains the definitions of the basic functors to be used, the introduction of the trace module, and the affine classification theorem. In §2, the Schlessinger deformation theory is applied to construct the versal deformation space and relate it to the structure constant tensor scheme C_n defined in §1. In the interest of accessibility to non-algebraic geometers, this section has been left optional. In §3 we translate the theoretical material in §2 into an implementable algorithm, and work out one example in detail. The more algebrogeometrical parts of the argument are relegated to Appendix 1. In §4 we return to algebraic geometry with the global classification theorem 3, and the calculation of birational parametrizations of the semi-rigid family in dim 4 and the Kronecker component. §5 contains the geometric interpretation of the construction of higher order neighborhoods of the versal deformation space, and Appendix 2 discusses the implementation of the algorithm in light of the theorem in Schaps [10] about the deformability of idempotents.

§1. The local theory of n-covers

Let Y be an integral, noetherian scheme over an algebraically closed field K. That is to say, Y is a topological space supporting a structure sheaf $\mathcal{O}_Y$ of K-algebras such that each $\mathcal{O}_Y(U)$ is a noetherian integral domain. Suppose that n is an integer, prime to the characteristic of K if $\operatorname{char} K \neq 0$. In algebraic geometry there is a very rich theory of 2-covers, i.e., schemes X with $p: X \to Y$ such that the structure sheaf $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$ modules which is locally free of rank 2. We wish to generalize some parts of this theory, specifically, those dealing with formal deformations and with classification data, to the case where the multiplication in the sheaf $\mathcal{O}_X$ is no longer required to be commutative.

The Noetherian and integrality hypotheses are intended to avoid difficulties in globalization, and will not enter into the local theory we develop now.

Definition. Let A be a commutative algebra over K, an algebraically closed field. Let n be prime to the characteristic of K if $\operatorname{char} K \neq 0$. An *affine n-cover B* of A will be an A-algebra B which is free as an A-module. The fibers $B \otimes_A A/m$ of B over the maximal ideals m of A will be called members of the *family* defined by B.

Since B is free as an A-module, the action of A on the identity of B determines an embedding $p: A \to B$. Since B is an A-algebra, the image of A lies in the center of B. Thus it is irrelevant whether we regard B as a left or a right A-module. We now fix this underlying A-module, so that we will be able to concentrate on the variations in the algebra structure.

DEFINITION. Let V_0 be a K-vector space of dimension n with designated basis $e_0,\dots,e_{n-1}$. For any K-algebra A, define

$$V(A)=V_0\otimes A.$$

If A is understood, we will simply write V for $V(A)$. Let $\tilde{V}_0\subset V_0$ be the $(n-1)$-dimensional vector space generated by $e_1,\dots,e_{n-1}$, and let $\tilde{V}(A)=\tilde{V}_0\otimes A$.

To assign a bilinear multiplication to V is to choose an A-module homomorphism

$$\rho\in \mathrm{Hom}_A(V,\mathrm{End}_A(V)).$$

Let $V^*=\mathrm{Hom}(V,A)$ be the dual module. Then for any A-module M,

$$\mathrm{Hom}_A(V,M)\tilde{\to} V^*\otimes M;$$

since V is a free A-module,

$$\begin{aligned}\mathrm{Hom}_A(V,\mathrm{End}_A(V))\tilde{\to}\ & V^*\otimes \mathrm{End}_A(V)\\ &= V^*\otimes \mathrm{Hom}_A(V,V)\\ &\tilde{\to} V^*\otimes V^*\otimes V.\end{aligned}$$

Let us continue to denote by $e_0,\dots,e_{n-1}$ the images of the basis elements of V_0 under the natural embedding of V_0 into the A-module V. Let $e_0^*,\dots,e_{n-1}^*$ be the elements of the dual basis for V^*, defined as functions on V by the property that $e_i^*(e_j)=\delta_{ij}$. Here, and throughout the paper, δ_{ij} represents the Kronecker δ-function, and equals 1 when $i=j$, but equals 0 when $i\neq j$. The tensors $e_i^*\otimes e_j^*\otimes e_p$ form a basis for $V^*\otimes V^*\otimes V$, and thus, via the isomorphism described in the previous paragraph, for $\mathrm{Hom}_A(V,\mathrm{End}_A(V))$. A tensor $\alpha=\sum a_{ij}^p e_i^*\otimes e_j^*\otimes e_p$ corresponds to a multiplication

$$e_i\cdot e_j=\sum a_{ij}^p e_p. \tag{1}$$

If $\rho\in\mathrm{Hom}_A(V,\mathrm{End}_A(V))$ is the corresponding homomorphism, we can think of $\rho(e_i)$ as being represented with respect to the given basis by a matrix $[a_{ij}^p]$ with rows indexed by p and columns indexed by j.

If we have an A-multiplication structure on V and we change the A-module basis, we get an isomorphic A-algebra. Conversely any isomorphism of A-multiplication structures on V is completely determined by the induced A-module isomorphism. Thus the isomorphism classes of A-algebra structures correspond to the orbits in $V^* \otimes V^* \otimes V$ of the action of elements $Q \in \mathrm{Aut}_A(V)$ via $Q^* \otimes Q^* \otimes Q$, where $Q^* \in \mathrm{Aut}_A(V^*)$ is the element dual to Q. If Q can be represented in a given basis by a matrix M then Q^* is represented in the dual basis by ${}^TM^{-1}$.

In order to make our A-multiplication structure associative and unitary, we need to restrict our attention to those tensors (a^t_{ij}) for which the corresponding multiplication is associative and has an identity. In fact, we will go further and require that the identity be the basis element e_0. In this latter restriction we are not following the standard modern treatment of algebra structures (Gabriel [2], Happel [3], Mazzola [7]), but the choice of identity is necessary for our method. Since we want to give the algebra conditions in a basis-free form, the only way to ensure the existence of an identity is to specify one fixed element to be the identity and specify that it multiplies the other elements of the algebra as an identity element should.

We wish to describe these conditions by polynomial equations on the coordinates (a^p_{ij}) of a tensor in $V^* \otimes V^* \otimes V$. The requirement that $e_0 \cdot e_i = e_i \cdot e_0 = e_i$ gives

$$a^p_{0j} = \delta_{pj} \quad \text{and} \quad a^p_{i0} = \delta_{ip}. \tag{2}$$

The associativity, $(e_ie_j)e_k = e_i(e_je_k)$, gives equations

$$\sum_{t=0}^{n-1} a^t_{ij}a^q_{tk} - \sum_{t=0}^{n-1} a^t_{jk}a^q_{it} = 0. \tag{3}$$

In order to free ourselves of the choice of a basis, we define a bilinear form $\langle\ ,\ \rangle: (V^* \otimes V^* \otimes V) \otimes (V^* \otimes V^* \otimes V) \to V^* \otimes V^* \otimes V^* \otimes V$,

$$\langle v_1^* \otimes v_2^* \otimes v_3, v_4^* \otimes v_5^* \otimes v_6\rangle = v_4^*(v_3)v_1^* \otimes v_2^* \otimes v_5^* \otimes v_6 - v_5^*(v_3)v_4^* \otimes v_1^* \otimes v_2^* \otimes v_6.$$

If α and β are two tensors, written as (a^t_{ij}) and (b^q_{tk}) with respect to some choice of basis $e_0, \ldots, e_m$ of V, then with respect to that same basis, $\langle \alpha, \beta \rangle$ will have coordinates

$$c^q_{ijk} = \sum_{t=0}^{n-1} (a^t_{ij}b^q_{tk} - a^t_{jk}b^q_{it}).$$

Equations (3) are then equivalent to the condition $\langle \alpha, \alpha \rangle = 0$.

Definition. We define an (affine) algebraic group to be a representable functor $\mathcal{G}$ from K-algebras to groups, as in [15]. The groups we need will all be algebraic subgroups of

$$\mathcal{G}(A)=\mathrm{Aut}_A(V)=\mathrm{Aut}_A(V_0\otimes A).$$

DEFINITION. If $Q\in\mathrm{Aut}_A(V)$, let $Q^*\in\mathrm{Aut}_A(V^*)$ be the dual automorphism $Q^*v^*(w)=v^*(Q^{-1}w)$. $\mathrm{Aut}_A(V)$ acts on any tensor algebra $(\otimes_r V^*)\otimes(\otimes_s V)$ via $(\otimes_r Q^*)\otimes(\otimes_s Q)$, where

$$(\otimes_r Q^*)\otimes(\otimes_s Q)(v_1^*\otimes\cdots\otimes v_r^*\otimes v_1\otimes\cdots\otimes v_s)$$
$$=Q^*v_1^*\otimes\cdots\otimes Q^*v_r^*\otimes Qv_1\otimes\cdots\otimes Qv_s.$$

Let $Q\circ\alpha$ denote this action when $\alpha\in V^*\otimes V^*\otimes V$.

LEMMA 1. *All* $\alpha\in V^*\otimes V^*\otimes V$ *with* $\langle\alpha,\alpha\rangle=0$ *form a subset of* $V^*\otimes V^*\otimes V$ *which is invariant under the action of* $\mathrm{Aut}_A(V)$.

PROOF. One can cite, as a one-line proof, the fact that the associativity of an algebra is independent of choice of basis. To give a direct proof via tensor algebras we note that $Q^*v^*(Qw)=v^*w$. Thus

$$\left\langle Q\circ\sum_{i,j,t=0}^{m}a_{ij}^{t}e_i^{**}\otimes e_j^*\otimes e_t,\ Q\circ\sum_{t',k,q=0}^{m}a_{t'k}^{q}e_{t'}^*\otimes e_k^*\otimes e_q\right\rangle$$
$$=\sum_{i,j,k,q=0}^{m}c_{ijk}^{q}Q\circ(e_i^*\otimes e_j^*\otimes e_k^*\otimes e_q).$$

If $\langle\alpha,\alpha\rangle=0$, then $\langle Q\circ\alpha,Q\circ\alpha\rangle=0$.

DEFINITION. Let the algebraic group of affine automorphisms of V be given by

$$\mathcal{A}(A)=\{Q\in\mathrm{Aut}_A(V)\mid Q(e_0)=e_0\}.$$

This algebraic group $\mathcal{A}$ contains a normal subgroup $\mathcal{T}$ of translations by e_0. That is to say, the elements of $\mathcal{T}(A)$ correspond to A-linear automorphisms of V given by sending e_i to $e_i-e_0a_i$, for $a_i\in A$, and $i=1,\dots,n-1$. The group $\mathcal{A}$ contains another algebraic subgroup $\tilde{G}$ such that $\tilde{G}(A)$ corresponds to the A-linear automorphisms of V which leave e_0 fixed and map the submodule $\tilde{V}$ onto itself. Then $\mathcal{A}$ is in fact a semidirect product of $\mathcal{T}$ by $\tilde{G}$. This decomposition can be checked by taking a matrix representation of $\mathcal{A}$ and writing the matrices in block form with respect to the decomposition of V as $e_0A\oplus\tilde{V}$.

The identity equations (2), written in basis free form, are given by

$$e_0\cdot v=v\cdot e_0=v,\qquad\text{for all } v\in V.$$

LEMMA 2. *This property is invariant under the action of* $\mathcal{A}(A)=\mathrm{Aff}\,\mathrm{Aut}_{\mathcal{A}}(V)$.

PROOF. After acting by $Q \in \mathrm{Aut}_A(V)$, this condition becomes

$$Q(e_0)\cdot Q(v) = Q(V)\cdot Q(e_0) = Q(v), \qquad \forall v \in V.$$

If $Q \in \mathscr{A}(A)$ then $Q(e_0) = e_0$, and since Q is a bijection, we get the desired condition.

DEFINITION. For any K-algebra A, let

$$\mathscr{C}_n(A) \subset V^* \otimes V^* \otimes V$$

be the set of all tensors satisfying conditions (2) and (3).

DEFINITION. Let $c = (c_{ij}^p)$ be a set of indeterminates for $i, j, p = 0, \ldots, n-1$. Let

$$R_n = K[c]/(\langle c, c\rangle, c_{0j}^p - \delta_{jp}, c_{i0}^p - \delta_{ip})$$

where δ is the Kronecker delta function; $C_n = \mathrm{Spec}(R_n)$ is called the variety of structure constants.

THEOREM 1. *$\mathscr{C}_n(A)$ is a representable functor, represented by R_n, and the isomorphism classes of affine n-covers correspond to the orbits of $\mathscr{C}_n(A)$ under the action of $\mathscr{A}(A)$.*

PROOF. We identify a tensor $\alpha \in \mathscr{C}_n(A)$ with the homomorphism $R_n \to A$ given by $c_{ij}^p \to a_{ij}^p$. The homomorphism is well-defined since the coordinates (a_{ij}^p) of α satisfy the defining equations of R_n. Conversely, any homomorphism of R_n into A determines a tensor $\alpha \in \mathscr{C}_n(A)$.

Any affine n-cover over A has an underlying A-module isomorphic to V. Thus in order to classify the isomorphism classes of affine n-covers, it suffices to consider only A-algebras B whose underlying A-module is V itself.

Suppose, first, that B and B' are isomorphic A-algebra structures on V, with structure constant tensors $\alpha = (a_{ij}^p)$ and $\alpha' = (a_{ij}'^p)$. The A-algebra isomorphism from B to B' induces an A-linear automorphism Q of V, sending the identity e_0 to e_0. Thus $Q \in \mathscr{A}(A)$. Let us represent the multiplication in B by $\cdot$ and the multiplication in B' by $*$. Since we have an A-algebra isomorphism, $e_i \cdot e_j = \Sigma a_{ij}^p e_p$ implies that $Q(e_i) * Q(e_j) = \Sigma a_{ij}^p Q(e_p)$. Thus we have two different representations for α' with respect to two different bases,

$$\alpha' = \sum a_{ij}'^p e_i^* \otimes e_j^* \otimes e_p$$

$$= \sum a_{ij}^p Q^*(e_i^*) \otimes Q^*(e_j^*) \otimes Q(e_p).$$

The second sum is, by definition, $Q \circ \alpha$, so α' lies in the orbit of α under the action of $\mathscr{A}(A)$.

Conversely, let α be any element of $\mathscr{C}_n(A)$, and Q any element of $\mathscr{A}(A)$. By Lemmas 1 and 2, $Q \circ \alpha$ is also an element $\alpha' \in \mathscr{C}_n(A)$. Let B, B' be the A-algebra structures on V corresponding to α and α'. The representation given above of α' in the basis $Q(e_0), \ldots, Q(e_{n-1})$ shows that the A-module homomorphism Q induces an isomorphism between B and B'.

For any affine n-cover, not necessarily with underlying A-module V, we choose an A-module isomorphism between its A-module and V in order to associate to it an orbit of structure constant tensors. If we had chosen a different A-module isomorphism, the two isomorphisms would have differed by an automorphism of V. Thus, by what we have just proven, both isomorphisms would have given the same orbit, and the mapping from isomorphism classes of affine A-covers to orbits of $\mathscr{C}_n(A)$ is thus well-defined and bijective. Q.E.D.

DEFINITION. The n-fold point is the affine scheme whose underlying algebra is the unique local radical square zero algebra

$$K[x_1, \ldots, x_{n-1}]/(x_1, \ldots, x_{n-1})^2.$$

REMARK 1. The n-fold point lies in the closure of every orbit of C_n as a specialization at $\lambda = 0$ of the family of automorphisms of V given by multiplying every element of $\tilde{V}$ by the non-zero scalar λ^{-1}. Thus the orbit of the n-fold point is the unique closed orbit in C_n.

REMARK 2. *The history of C_n.* C_n has been studied for over a hundred years, though not necessarily in the detail it deserves. We review what is known. The commutative algebras are always contained in a single irreducible component with an open orbit corresponding to the semisimple commutative algebra, which is a product of n copies of the underlying field K. For $n = 2$, C_2 is a plane containing a quadric which is the orbit of the 2-fold point. For $n = 3$, C_3 has two components, the commutative component of dimension 6 which is an affine space, and a second component of dimension 3, also an affine space, whose open orbit corresponds to the algebra of upper triangular 2×2 matrices. For $n = 4, 5$ it is known which orbits are contained in the closures of which other orbits and thus what are the irreducible components. For $n = 4$ there are five irreducible components (Gabriel [2]) and for $n = 5$ there are ten (Happel [3], Mazzola [7]). For $n = 6$ there is an old list of the algebras (Voghera [14]) probably containing errors and not organized by specialization properties.

The algorithms described in §3 of this paper are intended to permit a closer study of the sturcture of C_n as an algebraic scheme: the nature of its singularities and the way in which the orbits fit together. We hope to give a complete analysis of the algebraic structure of C_4 in another work, and in the meantime we will bring two examples from C_4 in the course of this paper.

We return now to the theoretical discussion. Although the functor $\mathscr{C}_n(A)$ is quite satisfactory for studying the local theory, the globalization in §4 requires that we allow variation in the underlying A-module, or, more precisely, in the module $\tilde{V}$ complementary to the identity. In order to choose such a complementary module in a canonical way, we will then resort to the trace map.

DEFINITION. Let $\mathscr{F}_l$ be the category of pairs (A, N) where A is a K-algebra and N is a free A-module of rank l. A morphism (f, θ) from (A, N) to (A', N') will consist of an A-algebra homomorphism $f\colon A \to A'$ and an A'-module isomorphism $\theta\colon N \otimes_A A' \xrightarrow{\sim} N'$. Every such morphism factors as

$$(A, N) \xrightarrow{(f,\mathrm{id})} (A', N \otimes_A A') \xrightarrow{(\mathrm{id},\theta)} (A', N').$$

DEFINITION. Let us denote $e_0K \otimes_K A$ by e_0A. For any A-module M of rank $n-1$, we denote by $\boldsymbol{M}$ the rank n A-module $e_0A \oplus M$. This determines a functor $H_n\colon \mathscr{F}_{n-1} \to \mathscr{F}_n$ given by $H_n(A, M) = (A, \boldsymbol{M})$. In particular, $H_n(A, \tilde{V}(A)) = (A, V(A))$.

DEFINITION. For any $(A, M) \in \mathscr{F}_{n-1}$, define

$$\mathscr{K}_{n-1}(M)_A \subset \boldsymbol{M}^* \otimes \boldsymbol{M}^* \otimes \boldsymbol{M}$$

to be the set of all tensors α satisfying the conditions

$$\langle \alpha, \alpha \rangle = 0, \tag{4}$$

(5) $e_0 \cdot v = v \cdot e_0 = v$, for all $v \in \mathbf{M}$, with respect to the multiplication given by α.

If $(f, \theta)\colon (A, M) \to (A', M')$ is a morphism in $\mathscr{F}_{n-1}$, then we get an induced morphism

$$(\boldsymbol{f}, \boldsymbol{\theta})\colon (\boldsymbol{M} \otimes \boldsymbol{A}')^* \otimes (\boldsymbol{M} \otimes \boldsymbol{A}')^* \otimes (\boldsymbol{M} \otimes \boldsymbol{A}') \to \boldsymbol{M}'^* \otimes \boldsymbol{M}'^* \otimes \boldsymbol{M}'.$$

If $\theta = \mathrm{id}$, we will denote this by $\boldsymbol{f}$, and if $f = \mathrm{id}$ we will denote this by $\boldsymbol{\theta}$. We set $\mathscr{K}_{n-1}(\theta)_f = (\boldsymbol{f}, \boldsymbol{\theta})$, restricted to $\mathscr{K}_{n-1}(M)_A$.

CLAIM. *$\mathscr{K}_{n-1}(M)_A$ is a functor.*

If $\theta\colon M_1 \to M_2$ is an A-module isomorphism, then $\boldsymbol{\theta}\colon \mathscr{K}_{n-1}(M_1)_A \to \mathscr{K}_{n-1}(M_2)_A$ is obviously a bijection. The functoriality of $\mathscr{K}_{n-1}(M)_A$ with respect to A-algebra homomorphisms $f: A \to A'$ follows from the functoriality of

$$\mathcal{H}_{n-1}(\bar{V}(A))_A = \mathcal{C}_n(A).$$

The verifications are standard.

In Section 4, we will generalize this functor so that A can be replaced by a sheaf $\mathcal{O}_Y$ of K-algebras, and M can be replaced by a locally free $\mathcal{O}_Y$ module $\mathcal{E}$ of rank $n-1$. However, in order to make $\mathcal{E}$ an object which can be recovered in a canonical way from the resulting algebra structure, we first replace the functor $\mathcal{H}$ by a subfunctor $\mathcal{H}^0$ defined by using the trace map.

Letting B be any n-cover of A with additive A-module V, the left multiplication gives an algebra representation

$$\rho: B \to \mathrm{End}_A(V).$$

If we follow ρ by the trace map, we get an A-module homomorphism

$$\mathrm{Tr}\,\rho: V \to A.$$

This determines a "trace zero" submodule $\bar{V} \subset V$, where $\bar{V} = \ker \mathrm{Tr}\,\rho$. If e_0 is the identity element, then $\mathrm{Tr}\,\rho(e_0) = n \neq 0$, so

$$V = e_0 A \oplus \bar{V}.$$

Any $v \in V$ has a translate

$$\pi(v) = v - \frac{1}{n}\mathrm{Tr}(\rho(v))\, e_0$$

which lies in $\bar{V}$, and any other translate of this element will *not* lie in $\bar{V}$. Thus given any basis $\{e_0, e_1, \dots, e_{n-1}\}$ there is a unique translation $T \in \mathcal{T}$ which maps each e_i for $i \neq 0$ into $\bar{V}$.

Definition. Let $\mathcal{H}^0_{n-1}(\bar{V})$ be the set of all tensors $\alpha \in \mathcal{H}_{n-1}(\bar{V})$ such that

$$\mathrm{Tr}\,\rho_\alpha \mid \bar{V} = 0, \tag{6}$$

ρ_α defined by the multiplicative structure determined by α. Let $\mathcal{C}^0_n(A)$ be the corresponding subset of $\mathcal{C}_n(A)$. Denote $\mathrm{Tr}\,\rho_\alpha \mid \bar{V}$ by $\mathrm{Tr}(\alpha)^{\bar{}}$.

Definition. Let $R^0_n = R_n / (\Sigma^n_{j=0} c^j_{ij})^n_{i=1}$, and let $C^0_n = \mathrm{Spec}(R^0_n)$.

Remark. In C^0_n the orbit of the n-fold point is a single point, and the dimension of every other orbit is $n-1$ less than the corresponding orbit in C_n. Thus, for example, C^0_4 has two components of dimension 4 and 1, intersecting in a single point.

LEMMA 3. *$\mathcal{K}^0_{n-1}(M)_A$ is a functor from $\mathcal{F}_{n-1}$ to ((sets)) and thus $\mathcal{C}^0_{n-1}(A)$ is a functor from K-algebras to ((sets)), represented by R^0_n.*

PROOF. As described above, any morphism (f, θ) in $\mathcal{F}_{n-1}$ induces a morphism $(f, \boldsymbol{\theta})$: $\mathcal{K}_{n-1}(M)_A \to \mathcal{K}_{n-1}(M')_{A'}$. Furthermore, this morphism factors naturally into a composition of $(f, \mathbf{id})$, denoted by f, with $(\mathbf{id}, \boldsymbol{\theta})$, denoted by $\boldsymbol{\theta}$. Thus it suffices to show that f and $\boldsymbol{\theta}$ map the trace zero structure constant tensors to trace zero tensors. We first consider f: $\mathcal{K}_{n-1}(M)_A \to \mathcal{K}_{n-1}(M \otimes_A A')_{A'}$. Since $\mathrm{Tr}(f(\alpha))^{\sim} = f(\mathrm{Tr}(\alpha))^{\sim}$, we see that $\mathrm{Tr}(\alpha)^{\sim} = 0$ implies that $\mathrm{Tr}(f(\alpha))^{\sim} = 0$, so $f(\mathcal{K}^0_{n-1}(M)_A) \subset \mathcal{K}^0_{n-1}(M \otimes_A A')_{A'}$. Now consider $\boldsymbol{\theta}$: $\mathcal{K}_{n-1}(M \otimes_A A')_{A'} \to \mathcal{K}_{n-1}(M')_{A'}$. Let $f_1, \ldots, f_{n-1}$ be a basis for $M \otimes_A A'$ as a free A'-module. A tensor α lies in $\mathcal{K}^0_{n-1}(M \otimes_A A')_{A'}$ if and only if its coordinates (a^p_{ij}) with respect to the basis $e_0, f_1, \ldots, f_{n-1}$ satisfy $\Sigma_{j=0}^{n-1} a^j_{ij} = 0$ for each $i = 1, \ldots, n-1$. $\boldsymbol{\theta}(\alpha)$ has the same coordinates with respect to the image basis e_0, $\theta(f_1), \ldots, \theta(f_{n-1})$, and since θ is an isomorphism, $\theta(f_1), \ldots, \theta(f_{n-1})$ generate M', so $\boldsymbol{\theta}(\alpha) \in \mathcal{K}^0_{n-1}(M')_{A'}$, as required.

$\mathcal{C}^0_n$ is the composition of two functors, and is thus itself a functor. The representability of $\mathcal{C}^0_n$ by R^0_n follows from the representability of $\mathcal{C}_n$ by R_n proven in Theorem 1 and from the trace zero condition on the coordinates given in the first paragraph of this proof.

LEMMA 4. *For $n > 2$, a tensor $\alpha \in \mathcal{K}_{n-1}$ is completely determined by its projection $\tilde{\pi}\alpha$ onto $\tilde{V}^* \otimes \tilde{V}^* \otimes \tilde{V}$.*

PROOF. The A-module homomorphism $\tilde{\pi}$: $V^* \otimes V^* \otimes V \to \tilde{V}^* \otimes \tilde{V}^* \otimes \tilde{V}$ is induced by the dual i^*: $V^* \twoheadrightarrow \tilde{V}^*$ of the injection i: $\tilde{V} \to V$, and by the projection $\tilde{\pi}$: $V \to \tilde{V}$. Thus it suffices to show that the elements a^0_{ij} for $i \cdot j \neq 0$ are completely determined by the elements a^t_{ij} for $i \cdot j \cdot t \neq 0$, since the coordinates of α for $i = 0$ or $j = 0$ are fixed by the identity condition (5). Since $\alpha \in \mathcal{K}_n(\tilde{V})$, it also satisfies $\langle \alpha, \alpha \rangle = 0$. In order to find a^0_{ij}, we look at $\langle \alpha, \alpha \rangle^k_{ijk}$ for any $k \neq 0, i$. We get an equation

$$\sum_{t=0}^{m} a^t_{ij} a^k_{tk} - a^t_{jk} a^k_{it} = 0.$$

Since $a^k_{0k} = 1$ and $a^k_{i0} = 0$ by (2), we have

$$\begin{aligned} a^0_{ij} &= -\left(\sum_{t=1}^{m} a^t_{ij} a^k_{tk} - a^t_{jk} a^k_{it} \right) \\ &= -\langle \tilde{\pi}\alpha, \tilde{\pi}\alpha \rangle^k_{ijk}. \end{aligned}$$

Q.E.D.

REMARK. If $\alpha \in \tilde{\mathcal{H}}_n(\tilde{V})$, then for $i \neq k$, $\langle \tilde{\pi}\alpha, \tilde{\pi}\alpha \rangle^k_{ijk}$ and $\langle \tilde{\pi}\alpha, \tilde{\pi}\alpha \rangle^i_{ijk}$ are independent of k and i respectively, and $\langle \tilde{\pi}\alpha, \tilde{\pi}\alpha \rangle^q_{ijk} = 0$ for other values $i, j, k, q = 1, \ldots, n$.

We now reformulate Theorem 1 in terms which will be appropriate for globalization.

DEFINITION. Let $\mathcal{D}_n$ be the category of triples (A, M, α) such that A is a K-algebra, M is a free A-module of rank $n-1$, and $\alpha \in \mathcal{H}^0_{n-1}(M)_A$. A morphism is a triple $(f, \theta, (f, \boldsymbol{\theta})): (A, M, \alpha) \to (A', M', \alpha')$, where $(f, \theta): (A, M) \to (A', M')$ is a morphism in $\mathcal{F}_{n-1}$, and $(f, \theta)(\alpha) = \alpha'$.

DEFINITION. For any n-cover B of A, let $E(B)$ be its trace zero module, and let

$$\alpha(B) \in \mathcal{H}^0_{n-1}(E(B))$$

be its structure constant tensor. Conversely, for any n-cover data triple $(A, M, \alpha) \in \mathcal{D}_n$, let $\boldsymbol{M}$ be the A-algebra with underlying A-module M and structure constant tensor α.

THEOREM 1'. *The isomorphism classes of affine n-covers are in one-to-one correspondence with the isomorphism classes of data triples (A, M, α).*

PROOF. Consider the functions

$$B \to (A, E(B), \alpha(B)),$$

$$\boldsymbol{M}_\alpha \leftarrow (A, M, \alpha).$$

The composition $B \to (\boldsymbol{E}(\boldsymbol{B}))_{\alpha(B)}$ is a natural isomorphism obtained by sending the identity element of B to $e_0 \in \boldsymbol{E}(\boldsymbol{B})$. Now consider the composition from $\mathcal{D}_n \to \mathcal{D}_n$. Since $E(\boldsymbol{M}_\alpha) = M$, and $\alpha(\boldsymbol{M}_\alpha) = \alpha$, this composition is actually the identity. We conclude that we have an equivalence of categories, and thus a one-to-one correspondence between isomorphism classes.

§2. Formal moduli of n-dimensional algebras

Let $\mathcal{L}$ be the category of local Artin algebras over the algebraically closed field K, and let $\hat{\mathcal{L}}$ be the category of K-algebras which are inverse limits of directed systems of elements in $\mathcal{L}$, all of which are complete local rings over K. A functor F on $\mathcal{L}$ is called prorepresentable, if there is an object R in $\hat{\mathcal{L}}$ such that F is isomorphic to $\mathrm{Hom}_K(R, \cdot)$. It is said to have a prorepresentable hull R if there is a morphism of functors $\mathrm{Hom}_K(R, A) \to F(A)$ which is surjective for all A, and an isomorphism for the tangent space.

Let B_0 be a K-algebra of dimension n.

DEFINITION. A *deformation* B of B_0 over a K-algebra A is an affine n-cover B over A, together with a given isomorphism $b \otimes_A K \xrightarrow{\sim} B_0$. Identify the underlying vector space of B_0 with V_0, and let α_0 be the structure constant tensor of B_0. Define

$$\mathscr{C}_{n,\alpha_0}(A) = \{\alpha \in \mathscr{C}_n(A) \mid \alpha \otimes_A K = \alpha_0\}.$$

$\mathscr{C}_{n,\alpha_0}(A)$ is acted on by the subgroup $\mathscr{A}_I(A)$ of $\mathscr{A}(A)$ consisting of all automorphisms $Q \in \mathscr{A}(A)$ such that $Q \otimes_A K = I_{V_0}$. The orbits of $\mathscr{C}_{n,\alpha_0}$ under the action of this group will consist of structure constants whose corresponding algebras are isomorphic not only as affine n-covers but also as deformations.

Extending the Schlessinger deformation theory to the case in which B_0 and B are not necessarily commutative, we define the deformation function $D_{B_0}(A)$ on $\mathscr{L}$ to be the set of isomorphism classes of deformations of B_0 over A. $D_{B_0}(A)$ can be identified with the set of orbits of $\mathscr{C}_{n,\alpha_0}(A)$ under the action of $\mathscr{A}_I(A)$. Let $K[\varepsilon]$, $\varepsilon^2 = 0$ be the two-dimensional algebra of the two-fold point.

THEOREM 2. *$\mathscr{C}_{n,\alpha_0}$ on $\mathscr{L}$ is a prorepresentable functor prorepresented by the completion R_{n,α_0} of R_n at α_0. D_{B_0} has a prorepresentable hull whose tangent space $D_{B_0}(K[\varepsilon])$ can be canonically identified with a quotient of $\mathscr{C}_{n,\alpha_0}(K[\varepsilon])$ by the orbit of α_0 under the action of $\mathscr{A}_I(K[\varepsilon])$.*

PROOF. The prorepresentability of $\mathscr{C}_{n,\alpha_0}$ on $\mathscr{L}$ is a direct consequence of the representability of $\mathscr{C}_n$ in the category of K-algebras. We have an isomorphism of functors taking $\alpha \in \mathscr{C}_n(A)$ to $\theta_\alpha \in \mathrm{Hom}_K^{\mathrm{alg}}(R_n, A)$. If $\alpha \in \mathscr{C}_{n,\alpha_0}(A)$ for $A \in \mathscr{L}$, then $\alpha \otimes_A K = \alpha_0$ implies that θ_α factors through the localization R_{n,α_0} of R_n at α_0. Furthermore, since A, being an Artin algebra, is complete, θ_α further factors through the completion $\hat{R}_{n,\alpha_0}$ of this local ring. Conversely, any algebra homomorphism θ in $\mathrm{Hom}(R_{n,\alpha_0}, A)$ determines $\alpha = (\theta_\alpha(c_{ij}^p))$. The mapping $\theta \to \alpha_0$ is one-to-one, since θ is completely determined by the images of the generators c_{ij}^p.

We wish now to identify $D_{B_0}(A)$ with the set of orbits of $\mathscr{C}_{n,\alpha_0}(A)$ under the action of $\mathscr{A}_I(A)$. We know from Theorem 1 that isomorphism classes of n-covers are in one-to-one correspondence with orbits of $\mathscr{C}_n(A)$ under the action of $\mathscr{A}$. Since we have chosen a fixed identification of the underlying K-vector space of V_0 with B_0, two structure constant tensors α and α' belonging to the same orbit of $\mathscr{A}(A)$ will give isomorphic deformations if and only if $\alpha \otimes_A K = \alpha_0 = \alpha' \otimes_A K$. However, since $\alpha' = Q \circ \alpha$, this is equivalent to requiring that $Q \otimes_A K = I_{V_0}$. Conversely, if $\alpha' = Q \circ \alpha$ for $Q \in \mathscr{A}_I(A)$, then $\tilde{V}_\alpha$ is isomorphic to $\tilde{V}_{\alpha'}$, not only as an n-cover but also as a deformation.

In order to show that D_{B_0} has a prorepresentable hull, we have to show that conditions (H1)–(H3) in Schlessinger [12] are satisfied. We write D for D_{B_0}. Let u': $A' \to A$ be a homomorphism of Artin algebras from $\mathscr{L}$, and let u'': $A'' \to A$ be a surjection from the same category. We consider the map

$$h: D(A' \times_A A'') \to D(A') \times_{D(A)} D(A'').$$

($H1$): We must first show that h is surjective. Let $\eta' \in D(A')$, $\eta \in D(A)$ and $\eta' \in D(A'')$ be isomorphism classes such that η' and η'' reduce to η after tensoring by A. We chose representative structure constant tensors $\alpha' \in \mathscr{C}_{n,\alpha_0}(A')$ and $\alpha'' \in \mathscr{C}_{n,\alpha_0}(A'')$ for η' and η''. Let $\beta' = \alpha' \otimes_{A'} A$ and $\beta'' = \alpha'' \otimes_{A''} A$. By hypothesis β' and β'' determine isomorphic deformations, and therefore lie in the same orbit of $\mathscr{C}_{n,\alpha_0}(A)$ under the action of $\mathscr{A}_I(A)$. Thus there is an A-module automorphism $Q \in \mathscr{A}_I(A)$ of $V(A)$ such that $\beta' = Q \circ \beta''$. Since $A'' \to A$ is surjective, Q can be lifted to an A-module automorphism $Q'' \in \mathscr{A}_I(A'')$. Set $\gamma'' = Q'' \circ \alpha''$. Since $Q'' \otimes_{A''} K = Q \otimes_A K = I$, we find that $\gamma'' \otimes_{A''} K = \alpha_0$, so γ'' and α'' lie in the same $\mathscr{A}_I(A'')$ orbit, showing that γ'' is an alternative representation of the isomorphism class η'', with the additional property that $\gamma'' \otimes_{A''} A = \beta'$. We now form the fiber product

$$\bar{\alpha} = \alpha' \times_{\beta'} \gamma''.$$

$\bar{\alpha}$ lies in $\mathscr{C}_{n,\alpha_0}(A' \times_A A'')$ because substitution of the coordinates of $\bar{\alpha}$ into the defining equations of R_{n,α_0} gives elements of $A' \times_A A''$ which vanish under both projections and thus vanish altogether. The algebra on $V(A' \times_A A'')$ with structure constant tensor $\bar{\alpha}$ is a representative of the element of $D(A' \times_A A'')$ which maps to η in $D(A')$ and η'' in $D(A'')$. This proves (H1), that the map is surjective.

($H2$): Let $K[\varepsilon]$, $\varepsilon^2 = 0$ be the algebra of the two-fold point. We need to show that if $A = K$ and $A'' = K[\varepsilon]$, then

$$h: D(A' \times_A A'') \to D(A') \times_{D(A)} D(A'')$$

is a bijection. We will in fact show this when $A = K$ and $A'' \to K$ is an arbitrary homomorphism of K-algebras, necessarily surjective. Let $\bar{A} = A' \times_K A''$. Since we have already shown that h is a surjection, it remains to show that it is one-to-one. More specifically, if $\bar{\eta} \in D(\bar{A})$, and $\bar{\alpha}$ is a representative of the corresponding orbit to $\mathscr{C}_{n,\alpha_0}(\bar{A})$, then we need to show that the orbit of $\bar{\alpha}$ is completely determined by the orbits of its images $\alpha' \in \mathscr{C}_{n,\alpha_0}(A')$ and $\alpha'' \in \mathscr{C}_{n,\alpha_0}(A'')$. Suppose $Q' \in \mathscr{A}_I(A')$ and $Q'' \in \mathscr{A}_I(A'')$. Since both Q' and Q'' reduce to I in $\mathscr{A}_I(K)$, we can form the fiber product $\bar{Q} = Q' \times_I Q''$ which will carry $\bar{\alpha}$ to $\bar{Q} \circ \bar{\alpha}$, a structure constant tensor which reduces to $Q' \circ \alpha'$ in $\mathscr{C}_{n,\alpha_0}(A')$ and to $Q'' \circ \alpha''$ in $\mathscr{C}_{n,\alpha_0}(A'')$.

$$(H3): \qquad \begin{aligned} \dim_K D_{B_0}(K[\varepsilon]) &< \dim \mathscr{C}_{n,\alpha_0}(K[\varepsilon]) \\ &\leqq \dim V(K[\varepsilon])^* \otimes V(K[\varepsilon])^* \otimes V(K[\varepsilon]) \\ &\leqq 8n^3. \end{aligned}$$

Thus the tangent space is finite dimensional.

It remains to demonstrate that there is a natural projection of $\mathscr{C}_{n,\alpha_0}(K[\varepsilon])$ onto $D_B(K[\varepsilon])$. Let m be the maximal ideal in a prorepresenting hull R of D_B, constructed as in Schlessinger's theorem, and let $\hat{m}$ be the maximal ideal of $\hat{R}_{n,\alpha_0}$, the prorepresenting ring for $\mathscr{C}_{n,\alpha_0}$. In the Schlessinger construction, the Zariski tangent space $(m/m^2)^*$ of R is canonically identified with $D(K[\varepsilon])$ and similarly $(\hat{m}/\hat{m}^2)^*$ is canonically identified with $\mathscr{C}_{n,\alpha_0}(K[\varepsilon])$. $D(K[\varepsilon])$ has a prorepresenting object

$$R\langle x_0, \ldots, x_{n-1}\rangle \Big/ \Big(x_i x_j - \sum c_{ij}^p x_p\Big)$$

which is the image of the identity under the surjection $\mathrm{Hom}_K(R, \cdot) \to D$. Any element $s \in (m/m^2)^*$ extends naturally to a functional s on m with kernel m^2, and determines a ring homomorphism $R \to K[\varepsilon]$ by sending each $c_{ij}^p \in m$ to $\varepsilon s(c_{ij}^p)$. This homomorphism then induces an element of $D(K[\varepsilon])$. Similarly, an element s of $(\hat{m}/\hat{m}^2)^*$ induces a homomorphism $\hat{R}_{n,\alpha_0} \to K[\varepsilon]$, which takes the representing structure constant tensor to $(a_{ij}^p + \varepsilon b_{ij}^p) \in \mathscr{C}_{n,\alpha_0}(K[\varepsilon])$, where (a_{ij}^p) is the structure constant tensor α_0 of B_0. $\mathscr{C}_{n,\alpha_0}(K[\varepsilon])$ has a vector space structure induced from that of $(\hat{m}/\hat{m}^2)^*$, and $D(K[\varepsilon])$ has a vector space structure induced from $(m/m^2)^*$. The elements of $\mathscr{C}_{n,\alpha_0}(K[\varepsilon])$ are all those tensors such that

$$\langle a_{ij}^p + \varepsilon b_{ij}^p, a_{ij}^p + \varepsilon b_{ij}^p\rangle = 0 \quad \text{and} \quad b_{i0}^i = b_{0j}^j = 0.$$

The elements of $D(K[\varepsilon])$ consist of the isomorphism classes of the corresponding algebras

$$K[\varepsilon]\langle x_0, \ldots, x_{n-1}\rangle \Big/ \Big(x_i x_j - \sum (a_{ij}^p - \varepsilon b_{ij}^p) x_p\Big).$$

The kernel of the corresponding homomorphism of vector spaces $\mathscr{C}_{n,\alpha_0}(K[\varepsilon]) \to D(K[\varepsilon])$ is precisely the set of all $\alpha' \in \mathscr{C}_{n,\alpha_0}(K[\varepsilon])$ such that the corresponding algebra is isomorphic to B_0, i.e., is the orbit of α_0 under the action of $\mathscr{A}_I(K[\varepsilon])$.

§3. Construction of the versal deformation space

Having proven the existence of the versal deformation space in §2, we now proceed to the more practical problem of constructing it. The method we give for constructing the tangent space has been implemented on a microcomputer, which is sufficiently powerful to handle algebras of low dimension ($n \leqq 10$) with sparse multiplication tables. For larger n the program would have to be transferred to a larger computer.

For those who skipped §2, we will give a brief description of the object we wish to construct. Let α be a structure constant tensor lying in the variety of structure constants. We wish to find all deformations of the corresponding algebra B_α, i.e., all algebras $B_{\alpha'}$ whose structure constant tensors lie "close" to α. We want, in fact, to find a certain number of parameters $T_1, \ldots, T_N$ such that the coordinates of α' will be polynomial functions of the coordinates of α and formal power series in the variables $T_1, \ldots, T_N$. If they are also *polynomial* functions in the T_i, then we say that the space is algebraicizable. We further require that all the $B_{\alpha'}$ be non-isomorphic to B_α, a condition which insures that our parameter space has been chosen as small as possible.

Except in exceptional cases (like algebra (24) in Mazzola [7]) the parameter space is a completion of a section of C_n transversal to the orbit of α under the group action on C_n. However, since C_n is defined by a large number of quadratic equations, its structure is difficult to compute. Thus instead of starting at C_n and cutting down, we start at α and build up. We first construct a minimal family of first order deformations; this is the object we call the tangent space. We then try to extend these first order deformations to higher orders.

Construction of the Tangent Space

We take the simplest possible Artin algebra with non-zero radical, $K[\varepsilon] \xrightarrow{\sim} K[t]/(t^2)$. A deformation of B_α over $K[\varepsilon]$ is an associative algebra with identity, $B_{\alpha'}$, such that $B_{\alpha'} \otimes_{K[\varepsilon]} K = B_\alpha$. This corresponds to a structure constant tensor $\alpha' = \alpha + \varepsilon\beta \in \mathscr{C}_n(K[\varepsilon])$.

LEMMA 6. *For each $\alpha \in \mathscr{C}_n(K)$, the tensors $\beta \in V^* \otimes V^* \otimes V$ for which $\alpha + \varepsilon\beta$ lies in $\mathscr{C}_n(K[\varepsilon])$ form a linear subspace W defined by the conditions*

$$(4') \qquad \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle = 0,$$

$$(5') \qquad \beta(\varepsilon_0 \otimes v) = \beta(v \otimes e_0) = 0 \qquad \textit{for } v \in V.$$

PROOF. From the definition of $\mathscr{C}_n$, the conditions for α, $\alpha + \beta$ to be in $\mathscr{C}_n(K)$ and $\mathscr{C}_n(K[\varepsilon])$ respectively are

$$(4) \qquad \langle \alpha, \alpha \rangle = 0; \qquad \langle \alpha + \varepsilon\beta, \alpha + \varepsilon\beta \rangle = 0,$$

and

$$(5) \quad \alpha(e_0 \otimes v) = (\alpha + \varepsilon\beta)(e_0 \otimes v) = v \text{ and } \alpha(v \otimes e_0) = (\alpha + \varepsilon\beta)(v \otimes e_0) = v.$$

Using the fact that $\varepsilon^2 = 0$, we get

$$\varepsilon(\langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle) = 0$$

and

$$\varepsilon\beta(v \otimes e_0) = \varepsilon\beta(e_0 \otimes v) = 0.$$

Multiplication by ε does not annihilate any elements of K, so we get (4′) and (5′) as desired. Q.E.D.

As a second stage we want to eliminate the trivial deformations determined by infinitesimal automorphisms.

DEFINITION. Two tensors β and β' are *equivalent* if $B_{\alpha+\varepsilon\beta}$ is obtained from $B_{\alpha+\varepsilon\beta'}$ by an automorphism $Q = I + \varepsilon M$ which is the identity when $\varepsilon = 0$.

REMARK. It will frequently be the case that β and β' are not equivalent in this sense, yet $B_{\alpha+\varepsilon\beta}$ and $B_{\alpha+\varepsilon\beta'}$ will be isomorphic via some automorphism Q which is *not* in the form $I + \varepsilon M$. A fuller discussion of this phenomenon is given before the example after Lemma 8.

LEMMA 7. *β and β' are equivalent if and only if they lie in the same coset of the vector space $U_\alpha \subset W_\alpha$ consisting of all tensors $\psi(M)\alpha$, where $\psi(M)$ is the operator defined on $V^* \otimes V^* \otimes V$ by the formula*

$$\psi(M) = I \otimes I \otimes M - I \otimes {}^T\!M \otimes I - {}^T\!M \otimes I \otimes I.$$

M ranges over all $n \times n$ matrices with first column zero.

PROOF. By the definition of equivalence given above, and by our discussion in §1 of the effect of an automorphism of B_α on α, we see that β and β' are equivalent if and only if there is a matrix $Q = I + \varepsilon M \in \operatorname{Aff}\operatorname{Aut} V(K[\varepsilon])$ such that

$$\alpha + \varepsilon\beta' = (Q^* \otimes Q^* \otimes Q)(\alpha + \varepsilon\beta).$$

Substituting for Q, and using the fact that $\varepsilon^2 = 0$, we have

$$\begin{aligned} Q^* \times Q^* \times Q &= {}^T(I+\varepsilon M)^{-1} \otimes {}^T(I+\varepsilon M)^{-1} \otimes (I+\varepsilon M) \\ &= (I - \varepsilon\, {}^TM) \otimes (I - \varepsilon\, {}^TM) \otimes (I + \varepsilon M) \\ &= I \otimes I \otimes I + \varepsilon(I \otimes I \otimes M - I \otimes {}^TM \otimes I - {}^TM \otimes I \otimes I) \\ &= I \otimes I \otimes I + \varepsilon\psi(M). \end{aligned}$$

Again applying $\varepsilon^2 = 0$, we get

$$\begin{aligned} (Q^* \otimes Q^* \otimes Q)(\alpha + \varepsilon\beta) &= (I \otimes I \otimes I)(\alpha + \varepsilon\beta) + \psi(M)(\alpha + \varepsilon\beta) \\ &= (\alpha + \varepsilon\beta) + \varepsilon\psi(M)\alpha. \end{aligned}$$

thus $\alpha + \varepsilon\beta$ is equivalent to $\alpha + \varepsilon\beta'$ if and only if $\beta' = \beta + \psi(M)\alpha$.

$I + \varepsilon M \in \operatorname{Aff}\operatorname{Aut}(V(K[\varepsilon]))$ if it is invertible and its first column is identical to the first column of I. $I + \varepsilon M$ is always invertible with inverse $I - \varepsilon M$. Thus the only condition on M is that its first column must be zero. The set of all such M form a subvector space of the set of $n \times n$ matrices, which we will denote by $\operatorname{aff}\operatorname{aut}(V(K))$. (The notation is intended to reflect the fact that it is the Lie algebra of the algebraic group $\operatorname{Aff}\operatorname{Aut}(V(\bar{\ }))$, though we make no explicit use of this fact.) Since ψ is a linear operator on the $n \times n$ matrices, $U_\alpha = \{\psi(M)\alpha \mid M \in \operatorname{aff}\operatorname{aut}(V(K))\}$ is a vector space. Thus β and β' are equivalent if and only if $\beta' \in \beta + U_\alpha$.

We wish to pick one element out of each coset. For our purposes the simplest way to do so is to choose a non-degenerate bilinear form $(\ ,\ )$ and consider the orthogonal complement $U_\alpha^\perp$ to U_α in W_α.

LEMMA 8. *Let $U_\alpha^\perp \subset V^* \otimes V^* \otimes V$ be the linear subspace consisting of all tensors $\beta \in V^* \otimes V^* \otimes V$ satisfying the following linear equations:*

(i) $\langle\alpha, \beta\rangle + \langle\beta, \alpha\rangle = 0$,

(ii) $\beta(v \otimes e_0) = \beta(e_0 \otimes v) = v$ *for all* $v \in V$,

(iii) $(\beta, \psi(M)\alpha) = 0$ *for all $n \times n$ matrices M with zero first columns.*

Then $U_\alpha^\perp$ contains exactly one β from each equivalence class of tensor β such that $B_{\alpha+\varepsilon\beta}$ is a deformation of B_α over $K[\varepsilon]$.

PROOF. Since $W_\alpha = U_\alpha \oplus U_\alpha^\perp$, each $\beta' \in W_\alpha$ has a unique projection to an element $\beta \in U_\alpha^\perp$. This β is, by Lemma 7, the unique element of the equivalence class of β' lying in W_α. By Lemma 6, the elements of W_α give all first order deformations.

REMARK. This $U_\alpha^\perp$ is isomorphic to the tangent space of the deformation functor, which consists of all first order deformations modulo infinitesimal isomorphism.

Implementation

Lemma 8 gives a large system of linear equations for the coordinates of β, with coefficients determined by the coordinates of α. The author has written a computer program which takes as input the non-zero entries in α, generates this system of equations, and solves it. In order for the program in its current forms to work effectively, α must be *sparse* in the sense that most of its coordinates are zero, but most algebras are given in a normal form for which this is true. Group algebras are a notable exception to this rule, but group algebras are semisimple and thus have no non-trivial deformations.

DEFINITION. An algebra is *rigid* if its orbit is dense in some component of C_n. A family is *semirigid* if the unions of the orbits of the algebras in an open subset of the family gives a dense subset of a component of C_n. The general algebra in the family will also be called *semirigid.*

Every component of C_n is the closure of the orbit of either a rigid algebra or a semirigid family. In dimension 4 there are four rigid algebras and one semirigid family, while in dimension 5 there are nine rigid algebras and one semirigid family. In dimensions $n \leqq 6$, the rigid algebras are semisimple or radical-square zero, except for the rigid algebra of upper triangular 3×3 matrices. The parameters of the semirigid families enter at the level of the radical-square.

If we can find a way to generate all candidates for rigid algebras and semirigid families, then they can be distinguished from non-rigid algebras by the tangent space of the deformation space, the object described in Lemma 8 and computed by the computer program. An ordinary algebra has first order deformations which are not equivalent but which *are* isomorphic. That is to say, given a structure constant tensor α, one can usually find deformations $\alpha+\varepsilon\beta$ and $\alpha+\varepsilon\beta'$ such that $B_{\alpha+\varepsilon\beta}$ is isomorphic to $B_{\alpha+\varepsilon\beta'}$ via some automorphism $Q\in \operatorname{Aff}\operatorname{Aut}(V(K[\varepsilon]))$, but Q is *not* of the form $I+\varepsilon M$. This happens whenever the orbit of α is contained in the closure of an orbit $O(\alpha')$ of larger dimension, for in that case the stabilizer of α is larger than the stabilizer of α'. Suppose we could find a rational curve of structure constant tensors $\tilde{\alpha}(t)=$

$\alpha + t\beta + t^2_\gamma \cdots$ whose generic element was in the orbit of α' but which equalled α for $t = 0$. Take Q to be an element of Aff Aut($V(K)$) which lies in the stabilizer of α but not in the stabilizer of $\bar{\alpha}(t)$ for general t. Q operating on $\bar{\alpha}(t)$ will give a new curve $\alpha + t\beta' + t^2_{\gamma'} \cdots$. If we consider $\bar{\alpha}(t)$ as a formal power series and mod out by t^2, we conclude that $Q \circ (\alpha + \varepsilon\beta) = \alpha + \varepsilon\beta'$. Thus we get isomorphic but non-equivalent first order deformations.

The only cases where this does not occur are cases where α is not contained in the closure of any other orbit. These are usually the cases of interest to us: the rigid and semirigid algebras.

The author has a student, Thierry Dana-Picard, working on the problem of generating and checking candidates for rigid algebras, using the program described above. The problem of locating semirigid algebras is somewhat more difficult.

Trace Zero Case

As was shown in §1, every algebra is isomorphic to a trace zero algebra. However, for calculating the tangent space it is important that the multiplication table be as sparse as possible, and thus it is not generally wise to translate the algebra to the trace zero form.

For local algebras the simplest form of the multiplication table often *is* in trace zero form, and if so we can take advantage of this fact. In place of all the equations $\langle B, \psi(M)\alpha\rangle$ given in Theorem 2, we use only the equations for which the first row of M, as well as the first column, is zero. This makes $I + \varepsilon M$ an element of $\tilde{G}(V(K[\varepsilon])$, and gives us the general infinitesimal automorphism leaving the image of α trace zero. In place of the m equations we have removed, we add n equations requiring β to be trace zero:

$$\beta^1_{i1} + \cdots + \beta^m_{im} = 0$$

for $i = 1, \ldots, m$. Thus both α and $\alpha + \varepsilon B$ lie in $\mathscr{C}^0_n(K)$, $\mathscr{C}^0_n(K[\varepsilon])$ respectively.

EXAMPLE 1. Every finite dimensional algebra with structure constant tensor $\bar{\alpha}$ has a linear deformation to the n-fold point given by

$$x_i \cdot x_j = \sum_{p=1}^{m} t\bar{a}^p_{ij} x_p + t^2 \bar{a}^0_{ij} x_0 \qquad \text{for } i, j > 0.$$

When t goes to zero, we get the trivial multiplication $x_i \cdot x_j = 0$ for all $i, j > 0$.

If all the $a^p_{ij} = 0$ for $i, j > 0$, then the equations $\langle \alpha, \beta\rangle + \langle \beta, \alpha\rangle = 0$ used in defining the tangent space are all trivial. Furthermore, all of the tensors $\psi(M)\alpha$ are equal to zero, for M mapping $\tilde{V} = \langle x_1, \ldots, x_m\rangle$ into itself. Thus the only non-trivial infinitesimal isomorphisms are the translations. Since $\Sigma a^{i}_{ij}=0$ for each $i>0$, we are in the trace zero case. Therefore we can assume that we use the equations appropriate to the trace zero case. Since these are the defining equations of the variety C^0_n of trace zero structure constants, we see that the versal deformation space of the n-fold point is algebraicizable. The formal parameter space is the vertex of C^0_n when regarded as a cone with respect to the action

$$\bar{a}^{p}_{ij}\to t\bar{a}^{p}_{ij} \qquad \text{for } i,j>0,$$

$$\bar{a}^{0}_{ij}\to t^2\bar{a}^{0}_{ij}.$$

A Versal Deformation Space Which is Not Irreducible

EXAMPLE 2. Find the versal deformation space of $B\tilde{\to}K[x,y]/(x^2,y^2)$ for $K=\mathbf{C}$. As a K-vector space, B has dimension 4, with basis $x_0=1$, $x_1=x$, $x_2=y$, $x_3=xy=yx$. Thus $B=B_\alpha$, where α is the tensor with $a^3_{21}=a^3_{12}=1$ and $a^{t}_{ij}=0$ for all other $i\cdot j\neq 0$. Since the calculation is somewhat complicated, and will in fact fill up most of the remainder of this section, we will divide it into subcategories.

EXAMPLE 2 (contd.). *Calculation of first order orbit of* α

Let us determine what are the various $\psi(M)\alpha$ for this α. We first note the general fact that if ν is the tensor of the n-fold point, then $\psi(M)\nu=0$ for all $M=\widetilde{\mathrm{gl}}(V(k))$, since the n-fold point has a presentation which is stable under the action of the automorphism group of $\tilde{V}$. We see that we need only consider $\psi(M)\tilde{\alpha}$, where $\tilde{\alpha}$ is obtained from α by replacing all entries with $i\cdot j=0$ by 0. Letting $E^{t}_{ij}=e^*_i\otimes e^*_j\otimes e_t$, we have $\tilde{\alpha}=E^3_{12}+E^3_{21}$. Let $E_{kl}=e^*_i\otimes e_k$.

We need to describe the action of $\psi(M)$ on a basis element E^{t}_{ij} of $V^*\otimes V^*\otimes V$. Letting $M=[m_{kl}]$ we have $M=\Sigma m_{kl}E_{kl}$,

$$M\cdot e_t=\sum_{q=1}^{m} m_{qt}e_q$$

and

$$e^*_i\cdot {}^TM=\sum_{q=1}^{m} m_{iq}e^*_q.$$

The summations begin with $q=1$ because $m_{kl}=0$ for all $k\cdot\ \ =0$. Since $\psi(M)=I\otimes I\otimes M-I\otimes{}^TM\otimes I-{}^TM\otimes I\otimes I$, we get

$$\psi(M)E^{t}_{ij}=\sum_{q=1}^{m} m_{qt}E^{q}_{ij}-m_{jq}E^{t}_{iq}-m_{iq}E^{t}_{qj}.$$

Thus in our case

$$(7)\qquad \psi(M)(E^3_{12}+E^3_{21})=\sum_{q=1}^{m} m_{q3}(E^q_{12}+E^q_{21})-m_{2q}(E^3_{1q}+E^3_{q1})-m_{1q}(E^3_{2q}+E^3_{q2}).$$

We first wish to calculate the dimension of the stabilizer of α. We note that m_{31} and m_{32} do not appear in (7), so E_{31} and $E_{32}\in \mathrm{stab}(\alpha)$. We also have $2E_{33}-E_{22}-E_{11}$, and $E_{22}-E_{11}$. These generate the entire stabilizer since $\dim \widetilde{\mathrm{gl}}(V(k))=9$, and we have five linearly independent tensors,

$$\psi(E_{13})\alpha = E^1_{12}+E^1_{21}-E^3_{23}-E^3_{32},\quad \psi(E_{21})\alpha = 2E^3_{11},\quad \psi(E_{33})\alpha = E^3_{12}+E^3_{21},$$

$$\psi(E_{23})\alpha = E^2_{12}+E^2_{21}-E^3_{13}-E^3_{31},\quad \psi(E_{12})\alpha = 2E^3_{22}.$$

EXAMPLE 2 (contd.). *Equations for the orthogonal subspace*

We wish to construct $R^{\perp}_{\alpha}$. Letting $\gamma=(c^t_{ij})$ with $c^t_{ij}=0$ for $i\cdot j=0$, and letting the inner product be the usual inner product $(\ ,\ )$ for a vector space over K with basis E^t_{ij}, we find that condition (iii) of Lemma 8 gives us five equations:

$$-c^3_{23}-c^3_{32}+c^1_{12}=-c^1_{12},\quad c^3_{11}=0,\quad c^3_{12}=-c^3_{21},$$

$$-c^3_{13}-c^3_{31}+c^2_{12}=-c^2_{21},\quad c^3_{22}=0.$$

EXAMPLE 2 (contd.). *Associativity equations*

We now apply condition (i), $\langle\alpha,\gamma\rangle+\langle\gamma,\alpha\rangle=0$,

$$\sum_{t=0}^{m} a^t_{ij}c^q_{tk}-a^t_{jk}c^q_{it}+c^t_{ij}a^q_{tk}-c^t_{jk}a^q_{it}=0.$$

We recall that $a^t_{ij}=0$ except for a^3_{12}, a^3_{21}, a^k_{0k}, a^i_{i0}, which equal 1. We have two symmetry transformations: interchanging 1 and 2, and switching the bottom two indices, while simultaneously interchanging i and k. We consider all possible equations according to different values of i, j, k and q.

Dividing into 15 cases according to the values of i, j and k, we get information from the following cases, after applying all symmetry transformations.

$j=k=3,\ i=2$: $c^0_{13}=c^0_{31}=c^2_{33},\ c^0_{23}=c^0_{32}=c^1_{33},\ c^0_{33}=0.$

$j=3,\ i=k=2,\ q=3$: $c^1_{23}=c^1_{32}$,
$c^2_{13}=c^2_{31}$.

$j=3,\ k=2,\ i=1,\ q=3$: $c^1_{13}=c^2_{32};\ c^2_{23}=c^1_{31}$.

$i=j=2,\ k=3,\ q=3$: $c^0_{22}=c^1_{23}=c^1_{32};\ c^0_{11}=c^2_{13}=c^2_{31}$.

$j=2,\ i=1,\ k=3,\ q=3$: $c^0_{12}=-c^3_{33}+c^2_{23};\ c^0_{21}=-c^3_{33}+c^1_{13}$.

$j=2,\ i=1,\ k=2,\ q=0$: $c^0_{23}=c^0_{32}=0$; $c^0_{13}=c^0_{31}=0$.

$j=2,\ k=2,\ i=1,\ q=3$: $c^3_{32}=c^3_{31}=c^3_{13}=c^3_{23}=0$.

$j=2,\ k=2,\ i=1,\ q=2$: $c^2_{32}+c^0_{12}=0$; $c^2_{23}+c^0_{21}=0$.

$j=2,\ k=1,\ i=1,\ q=3$: $c^3_{31}-c^3_{13}+c^2_{12}-c^2_{21}=0,$
$c^3_{32}-c^3_{23}+c^1_{21}-c^1_{12}=0.$

$j=2,\ k=1,\ i=1,\ q=1$: $c^1_{31}-c^1_{13}+c^0_{12}-c^0_{21}=0,$
$c^2_{32}-c^2_{23}+c^0_{21}-c^0_{12}=0.$

EXAMPLE 2 (contd.). *Solving the system*

Adding on the trace criterion, we have $c^1_{11}+c^2_{12}+c^3_{13}=0$. Since $c^3_{13}=0$ we get $c^2_{12}=-c^1_{11}$. Similarly $c^1_{21}=-c^2_{22}$. $c^1_{31}+c^2_{32}+c^3_{33}=2c^3_{33}=0$ so $c^3_{33}=c^1_{31}+c^2_{32}=0$.

Finally, we add the equations obtained from $\psi(M)\alpha$, now simplified by the substitution

$$c^3_{23}=c^3_{32}=c^3_{13}=c^3_{31}=0 \text{ above.}$$

The end result of combining all the linear conditions is the following tensor:

$$\overset{t=0}{\begin{bmatrix} a' & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix}} \quad \overset{t=1}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & e & a \\ 0 & a & 0 \end{bmatrix}} \quad \overset{t=2}{\begin{bmatrix} e' & 0 & a' \\ 0 & 0 & 0 \\ a' & 0 & 0 \end{bmatrix}} \quad \overset{t=3}{\begin{bmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$a=c^1_{23}=c^1_{32}=c^0_{22},$$

$$a'=c^2_{13}=c^2_{31}=c^0_{11};$$

$$e=c^1_{22},$$

$$e'=c^2_{11},$$

$$f=c^3_{21}=-c^3_{12}.$$

The tangent space of $\operatorname{Spec} R^{\perp}_{\alpha}$ has dimension 5.

EXAMPLE 2 (contd.). *Second order deformations*

Applying a computer implementation of the method in Appendix 1 to calculating the second order deformation produces equations

$$a\cdot f=a'\cdot f=e\cdot f=e'\cdot f=0$$

for the parameters. Thus the parameter space is reducible, consisting of one component satisfying $a = a' = e = e' = 0$ and one component satisfying $f = 0$. As explained in Appendix 1, there are many possible deformation spaces for this parameter space. It suffices to give one example:

$$x_1^2 = a' + ex_2,$$

$$x_2^2 = a + ex_1,$$

$$x_1x_2 = (1+f)x_3,$$

$$x_2x_1 = (1-f)x_3,$$

$$x_1x_3 = a'x_2 + ae' + ee'x_1,$$

$$x_3x_1 = a'x_2 + ae' + ee'x_1,$$

$$x_2x_3 = ax_1 + a'e + ee'x_2,$$

$$x_3x_2 = ax_1 + a'e + ee'x_2.$$

EXAMPLE 2 (contd.). *Interpretation and summary*

Finally, we consider the completion of the process from second order to power series. We calculate the two components separately:

(A) *The non-commutative component.* The given deformation is non-commutative only when $f \neq 0$, in which case

$$a = a' = e = e' = 0.$$

Thus we have a single parameter and every product is zero except for

$$x_1x_2 = (1+f)x_3,$$

$$x_2x_1 = (1-f)x_3.$$

Every product $(x_ix_j)x_k$ and $x_i(x_jx_k)$ for $i, j, k \geqq 1$ is zero, so all associativity conditions are fulfilled. Thus the deformation extends to all orders, and we get a linear, one-dimensional parameter space with parameter f. This is a semirigid deformation since two such algebras for f and f' are isomorphic if and only if $f' = -f$ or $f' = f$.

(B) *The commutative component.* In the second component, $f = 0$. We then have $x_1x_2 = x_3 = x_2x_3$ and the deformation is commutative, since we also have $x_1x_3 = x_3x_1$ and $x_2x_3 = x_3x_1$. Let $x_1 = x$, $x_2 = y$, and $x_3 = xy$. The general deformation given above is then entirely determined by the commutativity relations, the associativity relations, and the two equations

$$x^2 = a' + e'y,$$

$$y^2 = a + ex.$$

For example,

$$\begin{aligned} x_1x_3 &= x_1(x_1x_2) \\ &= (x_1^2)x_2 \\ &= (a' + e'x_2)x_2 \\ &= a'x_2 + e'x_2^2 \\ &= a'x_2 + e'(a + ex_1) \\ &= a'x_2 + e'a + e'ex_1. \end{aligned}$$

The algebra we are deforming is a complete intersection and thus its first order deformation can be lifted to all orders, giving a smooth parameter space of the same dimension. In this case the dimension is four and the parameters are a, a', e, e'. Furthermore, since we do not need power series to describe the deformation, the parameter space can be algebraicized.

In summary, the general versal deformation space of the algebra

$$K[x_1x_2x_3]/(x_1^2, x_2^2, x_1x_2 - x_3, x_1x_3, x_2x_3, x_3^2)$$

consists of a one-dimensional semirigid non-commutative component and a four-dimensional component which is the commutative versal deformation space of the codimension 2 algebra

$$K[x, y]/(x^2, y^2).$$

Completion and Algebraicization

We now take up the second stage of the construction — passing from the tangent space to a complete local ring, and if possible, to an algebraic ring. We fix the dimension n, and denote the representing ring of Theorem 1 by R, so that the notation R_p can be used as in Schlessinger's paper. We take a basis $\beta^1, \dots, \beta^r$ to the tangent space constructed above, and choose r indeterminants $T_1, \dots, T_r$. We let

$$\alpha_2 = \alpha + T_1\beta^1 + \cdots + T_r\beta^r$$

regarded as a structure constant tensor over $R_2 = K[T_1, \dots, T_r]/(T)^2$, i.e. $\alpha_2 \in C_n(R_2)$. For each $l = 3, 4, \dots$ let $S_l = K[[T_1, \dots, T_r]]/(T)^l$, the truncated power series ring. The essential content of Theorem 1 in §2 is that we can construct an inverse sequence $R_{i+1} \to R_i$ of rings, each R_i a quotient of S_i, and a sequence of elements $\alpha_i \in C_n(R_i)$ with the following properties:

(1) $\alpha_{i+1} \otimes_{R_{i+1}} R_i = \alpha_i$.

(2) If A_i is any quotient of S_i, and γ_i is any structure constant tensor in $\mathscr{C}_n(A_i)$ for which $\gamma_i \otimes_{A_i} K = \alpha$, then there is a homomorphism θ: $R_i \to A_i$ such that $\gamma'_i = \alpha_i \otimes_{R_i} A_i$ lies in the same orbit as γ_i. (The theorem itself would only give the algebras B_{α_i}. We apply Theorem 1 of §1 to get the structure constant tensor α_i.)

Definition. Letting R_∞ be the inverse limit of this sequence of rings and α_∞ the inverse limit of the sequence of structure constants, we call Spec(R_∞) the parameter space and B_{α_∞} the deformation algebra of B_α.

We may write $\alpha_\infty = \alpha + \Sigma_{l=1}^r T_l \beta^l + \Sigma_{l,l'=1}^r T_l T_{l'} \beta^{l,l'} + \cdots$. If we want to attempt a construction of R_∞ and α_∞, we do so by successively constructing $R_{p+1} = S_{p+1}/J_{p+1}$ and α_{p+1}. Given R_p and α_p, we set

$$\alpha_{p+1} = \alpha_p + \sum_{|\mu|=p} T^\mu B^\mu,$$

for the set of multi-indices $\mu = (\mu_1, \ldots, \mu_r)$, $T^\mu = T_1^{\mu_1} \cdots T_r^{\mu_r}$. Substituting in the formula from Theorem 1,

$$\langle \alpha_{p+1}, \alpha_{p+1} \rangle \equiv 0 \bmod T^{p+1},$$

we have

$$0 \equiv \langle \alpha_{p+1}, \alpha_{p+1} \rangle \equiv \langle \alpha_p, \alpha_p \rangle + \sum_{|\mu|=p} (\langle \alpha, \beta^\mu \rangle + \langle \beta^\mu, \alpha \rangle) T^\mu.$$

Since $\langle \ , \ \rangle$ has n^4 components, this is a large system of equations.

From Schlessinger's theorem on functors of Artin rings it is known that there is a minimal ideal J^{p+1} for which this system is solvable.

In Appendix 1 we give the construction of this ideal in the particular situation of deformations of Artin algebras. This too has been partially implemented for a microcomputer. Modulo this ideal we may choose a set M of μ such that $\{T_\mu \mid \mu \in M\}$ forms a basis for the vector space of monomials of degree p modulo J^{p+1}, and we may assume that $\beta^\mu = 0$ except for $\mu \in M$. For each $\mu \in M$ we get a system of equations for β^μ. Solving these equations for the various components $(\beta_{ij}^p)^\mu$ of β^μ, we then have

$$\alpha_{p+1} = \alpha_p + \sum_{\mu \in M} \beta^\mu T^\mu.$$

Continuing in this way, we construct R_∞ and α_∞.

§4. Glueing

We now globalize the results in §1, producing data triples which determine the global n-covers of a (commutative) integral scheme of finite type Y over K. By discussing the cases of double, triple, and quadruple covers, we then show how the information about deformations of algebras determined through the techniques of §2 and §3 can be used to make this classification concrete.

Recall that our ground field K is algebraically closed, of characteristic zero or prime to n. By definition of an integral scheme ([5], p. 82), Y has a covering by affine open sets $Y = \mathrm{Spec}(A_i)$, with A_i a finitely-generated K-algebra which is an integral domain. Let $\mathcal{O}_Y$ be the structure sheaf of Y, with $\mathcal{O}_Y(Y_i) = A_i$. In this very classical situation, the closed points of Y_i can be identified with a subset of an affine K-space which is closed and irreducible in the Zariski topology, and A_i can be identified with the regular functions on this algebraic set.

Definition. A (global) n-cover of $\mathcal{O}_Y$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-algebras which is a locally free $\mathcal{O}_Y$-module of rank n.

We will follow the notational conventions in Hartshorne [5]. Elements of the $\mathcal{O}_Y(U)$-algebras $\mathcal{F}(U)$ will be called *sections* of $\mathcal{F}$ over U. If $V \subset U$ are open sets, the restriction homomorphism from $\mathcal{F}(U)$ to $\mathcal{F}(V)$ will be denoted by ρ_{UV}, and if $s \in \mathcal{F}(U)$ we will sometimes denote $\rho_{UV}(s)$ by $s|_V$. An element s of $\mathcal{F}(Y)$ will be called a global section. Suppose $\{U_i\}_{i\in I}$ is an open cover of Y, and $\mathcal{H}_i$ is a sheaf on U_i for each $i \in I$ such that for each i, j there is an isomorphism of sheaves $\varphi_{ij}: \mathcal{H}_i|_{U_i\cap U_j} \xrightarrow{\sim} \mathcal{H}_j|_{U_i\cap U_j}$ such that:

(1) For each i, $\varphi_{ii} = \mathrm{id}$.

(2) For each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Then there exists a unique sheaf $\mathcal{H}$ on Y, together with isomorphisms $\psi_i: \mathcal{H}_{U_i} \xrightarrow{\sim} \mathcal{H}_i$, such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$. We will say that $\mathcal{H}$ is obtained by glueing the sheaves $\mathcal{H}_i$ ([5], p. 69). Although this result is given in Hartshorne [5] for sheaves of abelian groups, it is true as well for sheaves of sets.

We wish now to establish the global analogues of the two functors given in Theorem 1′. We start first with a global n-cover $\mathcal{F}$ of $\mathcal{O}_Y$, and construct a data triple. Let $\{U_i\}$ be an open covering of Y such that each $\mathcal{F}(U_i)$ is a free $\mathcal{O}_Y(U_i)$-module. Let $\mathcal{E}_i(U) = E(\mathcal{F}(U))$ for $U \subset U_i$ be the sheaf of trace zero modules of the $\mathcal{O}_Y(U)$-algebras $\mathcal{F}(U)$. Since $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-algebras we have $\mathcal{O}_Y(U_i \cap U_j)$-algebra isomorphisms

$$\varphi_{ij}: \rho_{U_i(U_i \cap U_j)}(\mathcal{F}(U_i)) \xrightarrow{\sim} \rho_{U_j(U_j \cap U_i)}\mathcal{F}(U_j),$$

and these φ_{ij} satisfy conditions (1) and (2). Since, by Lemma 3 of §1, the trace zero module is preserved by algebra isomorphism, the system of functions $\{\varphi_{ij}\}$ is just what is needed to glue the sheaves $\mathcal{E}_i$ together into a sheaf of $\mathcal{O}_Y$-modules which is locally free of rank $n-1$.

Suppose now that $\mathcal{E}$ is any locally free sheaf of $\mathcal{O}_Y$-modules of rank $n-1$, and that $\mathcal{E}^*$ is its dual. $\mathcal{E}^* \otimes \mathcal{E}^* \otimes \mathcal{E}$ is also a sheaf of $\mathcal{O}_Y$-modules, with restriction functions $\boldsymbol{\rho}_{UV}$ and transition functions $\boldsymbol{\varphi}_{ij}$ induced from those of $\mathcal{E}$. For any $U \subset U_i$, define

$$(\mathcal{K}^0_{n-1} \circ \mathcal{E})_i(U) = \mathcal{K}^0_{n-1}(\mathcal{E}_i(U)) \subset (\mathcal{E}^*_i \otimes \mathcal{E}^*_i \otimes \mathcal{E}_i)(U).$$

The restriction functions $\boldsymbol{\rho}_{UV}$ make this a sheaf of (affine algebraic) sets, and the transition functions $\boldsymbol{\varphi}_{ij}$ permit them to be glued into a sheaf $\mathcal{K}^0_{n-1} \circ \mathcal{E}$ on Y.

Let us return to the case where $\mathcal{E}_{\mathcal{F}}$ was obtained as the trace zero module of a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-algebras. For any i, and any $U \subset U_i$, define

$$\alpha_{\mathcal{F},i} = \alpha(\mathcal{F}(U)) \in \mathcal{K}^0_{n-1}(\mathcal{E}_i(U)).$$

If j is another index such that $U \subset U_j$, then clearly

$$\boldsymbol{\varphi}_{ij}(\alpha_{\mathcal{F},i}(U)) = \alpha_{\mathcal{F},j}(U).$$

Thus the collection $\alpha_{\mathcal{F},i}(U_i)$ of local sections of $\mathcal{K}^0_{n-1} \circ \mathcal{E}_{\mathcal{F}}$ determines a global section $\alpha_{\mathcal{F}}(Y)$ of $\mathcal{K}^0_{n-1} \circ \mathcal{E}_{\mathcal{F}}$.

Definition. A global n-cover data triple $(\mathcal{O}_Y, \mathcal{E}, \alpha)$ consists of (i) the structure of sheaf $\mathcal{O}_Y$ of a commutative integral scheme Y of finite type, (ii) a locally free $\mathcal{O}_Y$-module, and (iii) a global section α of $\mathcal{K}^0_{n-1} \circ \mathcal{E}$. They form a category $\bar{\mathcal{D}}_n$ in which the morphisms are given by triples $(f, \theta, (\boldsymbol{f}, \boldsymbol{\theta}))$ as in the category $\mathcal{D}_n$ of affine data triples.

Theorem 3. *The isomorphism classes of global n-covers is in one-to-one correspondence with the isomorphism classes of data triples $(\mathcal{O}_Y, \mathcal{E}, \alpha)$.*

Proof. We construct an equivalence of categories. We have already defined the functor

$$\mathcal{F} \rightarrow (\mathcal{O}_Y, \mathcal{E}_{\mathcal{F}}, \alpha_{\mathcal{F}}),$$

and now we reverse the process. Given a data triple $(\mathcal{O}_Y, \mathcal{E}, \alpha)$, we let $\boldsymbol{\mathcal{E}}$ be the locally free sheaf $\mathcal{O}_Y \oplus \mathcal{E}$. Let $\{U_i\}$ be an open cover of Y such that $\mathcal{E}(U_i)$ is a free $\mathcal{O}_Y(U_i)$-module. Let $\mathcal{F}_i$ be the sheaf of $\mathcal{O}_{U_i}$ algebras on U_i determined by

$\alpha(U_i)$. The transition functions

$$\varphi_{ij} : \tau_{U_i(U_i \cap U_j)}(\mathscr{E}(U_i)) \to \rho_{U_j(U_j \cap U_i)}(\mathscr{E}(U_j))$$

induce corresponding transition functions $\mathrm{id} \oplus \varphi_{ij}$ on the collection $\{\mathscr{F}_i\}$ of sheaves of modules, which satisfy conditions (1) and (2) for glueing of sheaves. Furthermore, since the corresponding induced isomorphisms

$$\varphi_{ij} : \mathscr{K}^0_{n-1}(\rho_{U_i(U_i \cap U_j)}(\mathscr{E}(U_i))) \to \mathscr{K}^0_{n-1}(\rho_{U_j(U_j \cap U_i)}(\mathscr{E}(U_j)))$$

carry the local section $(\alpha|_{U_i})|_{U_i \cap U_j}$ to $(\alpha|_{U_j})|_{U_j \cap U_i}$, we conclude that each $\mathrm{id} \oplus \varphi_{ij}$ is actually an isomorphism of algebras. Thus there is a unique locally free sheaf $\mathscr{F}$ of $\mathcal{O}_Y$-algebras of rank n obtained by glueing together the $\mathscr{F}_i$. We will denote it by $\mathfrak{E}_\alpha$. From the local isomorphisms cited in Theorem 1′, we see that the functor $\mathscr{F} \to (\mathfrak{E}_{\mathscr{F}})$ is a natural isomorphism, as is the fuunctor $(\mathcal{O}_Y, \mathscr{E}, \alpha) \to (\mathcal{O}_Y, \mathscr{E}_{\mathscr{F}}, \alpha_{\mathscr{F}})$, where $\mathscr{F} = \mathfrak{E}_\alpha$. Q.E.D.

The interest in the global version of this theorem lies in the possibility of constructing $\mathscr{K}^0_{n-1}(\mathscr{E})$ as a functor of $\mathscr{E}$. $\mathscr{K}^0_{n-1}(\mathscr{E})$ can actually be endowed with the structure of a (commutative) scheme over Y, but the structure of this scheme varies as $\mathscr{E}$ varies. To make this clear, we will first review the classical case $n = 2$ and then the case $n = 3$ from Miranda [8] and Miranda and Teicher [9].

$n = 2$: These are the classical double covers of algebraic geometry. $\mathscr{E}$ is a line bundle over Y. Let U be an open set on which $\mathscr{E}(U)$ is free, and let e_1 be a basis of the one-dimensional $\mathcal{O}_Y(U)$-module $\mathscr{E}(U)$. $e_1 \cdot e_1 = a^0_{11}e_0 + a^1_{11}e_1$, giving a minimal polynomial for e_1. Since e_1 must have trace zero and $e_1 \cdot e_0 = 0 \cdot e_0 + 1 \cdot e_1$, we must have $a^1_{11} = 0$. Thus, $\alpha(U)$ is completely determined by a morphism $\mathscr{E}(U) \otimes \mathscr{E}(U) \to \mathcal{O}_Y(U)$. We can thus identify $\alpha(U)$ with an element of $\mathscr{E}(U)^* \otimes \mathscr{E}(U)^*$, which we write classically as $\mathscr{E}^{-2}(U)$. This construction sheafifies, so we have the classical result that a double cover of Y is equivalent to a line bundle $\mathscr{E}$ on Y and a global section of $\mathscr{E}^{-2}$.

$n = 3$: We have taken all our covers over integral schemes, so each cover lies generically in a fixed irreducible component of C_n. For $n = 3$, there are two such components, one corresponding to commutative algebras, and one generically isomorphic to the algebra of upper triangular matrices.

In Miranda [8], he identifies global sections of $\mathscr{K}^0_2(\mathscr{E})$ lying in the commutative component with global sections of the sheaf $S_3(\mathscr{E})$, the symmetric algebra on $\mathscr{E}$ generated by $\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{E}$ modulo all commutativity relations on the tensor product, a space of dimension four. In Miranda and Teicher [9] they identify the global sections of the non-commutative algebras with global sections of $\mathscr{E}^*$.

$n=4$: *The semirigid family*. One of the five components of C_4^0 is that containing the semirigid family of algebras, which appeared in the example of §3 as the non-commutative component of the deformation space of $K[x, y]/(x^2, y^2)$. We will construct a parametrization of this component W, that is to say, we will find a birationally equivalent scheme N whose structure is easily described. C_n^0 can be regarded as a subscheme of $V_0^* \otimes V_0^* \otimes V_0$. Furthermore, as we showed in Lemma 4, each trace zero structure constant tensor α is completely determined by its projection $\tilde{\alpha}$ onto $\tilde{V}_0^* \otimes \tilde{V}_0^* \otimes \tilde{V}_0$. To simplify notation we will denote $\tilde{V}_0$ by E. Let s_0 be the unique point of W corresponding to $\tilde{\alpha}=0$. Let $Gr_d(E)$ denote the Grossmanian variety of d-dimensional subspaces of E. An examination of the deformation chart in Gabriel [2] shows that at every other point of W, we have a morphism L: $W \to Gr_1(E)$, such that $L(\alpha)$ is the one-dimensional subspace of E corresponding to the radical squared, J^2, in the multiplication determined by α. $L(\alpha)$ is also in the kernel of multiplication from left and right, so that

$$\tilde{\alpha} \in (E/L(\alpha))^* \otimes (E/L(\alpha))^* \otimes L(\alpha).$$

Conversely, for any $L \in Gr_1(E)$, every tensor in $(E/L)^* \otimes (E/L)^* \otimes L$ determines an associative multiplication structure, since $J^3=0$ and thus the associativity relations are void. Let N be the fiber bundle over $Gr_1(E) \tilde{\to} \mathbf{P}^2$ with affine fiber

$$F(L) = (E/L)^* \otimes (E/L)^* \otimes L$$

and let N_0 be the zero section. Then $W - \{W_0\} \tilde{\to} N - N_0$. Furthermore, the embedding of $N \to E^* \otimes E^* \otimes E$ induced by the embedding $(E/L)^* \to E^*$ carries N_0 to W_0. Thus the birational correspondence $N \to W$ is in fact a morphism, blowing up the point W to $\mathbf{P}^2$. The dimension of N, and thus of W, is $\dim \mathbf{P}^2 + (\dim(E/L))^2 \dim L = 6$. Since W contains a one-parameter family of orbits, each orbit has dimension 5. The automorphism group $\tilde{\mathcal{G}}(K)$ has dimension 9, so that the stabilizer has dimension 4, as calculated in the literature. For a fixed element L of $Gr_1(E)$, the affine space $(E/L)^* \otimes (E/L)^* \otimes L$ can be identified with 2×2 quadratic forms M, with automorphisms Q acting by ${}^TQMA$. If

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

then the orbits can be parameterized by the invariant $(B-C)^2/AD-BC$ for $AD-BC \neq 0$.

In order to sheafify this construction, we define a fiber bundle $\mathcal{N}(\mathscr{E})$ whose fiber at a closed point of Y is isomorphic to N. For $U \subset U_i$, a section $\beta(U)$ of $\mathcal{N}(\mathscr{E}(U))$ is an element $(L(\beta), \tilde{\beta})(U)$ with $L(\beta)(U) \in Gr_1(\mathscr{E}(U))$ and

$$\tilde{\beta}(U) \in (\mathscr{E}(U)/L(\beta)(U))^* \otimes (\mathscr{E}(U)/L(\beta)(U))^* \otimes L(\beta)(U).$$

The canonical mapping $\tilde{\beta}(U) \to \tilde{\alpha}(U) \in \mathscr{E}(U)^* \otimes \mathscr{E}(U)^* \otimes \mathscr{E}(U)$ induces a birational morphism from $\mathcal{N}(\mathscr{E})$ onto $\mathscr{W}(\mathscr{E})$, the component of $\mathscr{K}^0_{n-1}(\mathscr{E})$ containing the semirigid family.

$n \geqq 4$: *The Kronecker component.* This is a direct generalization of the non-commutative component in the $n = 3$ case given in Miranda and Teicher [9]. For any n, the Kronecker algebra of dimension n is the algebra with two idempotents e_1 and e_2, and a square zero radical J such that $J = e_1 J e_2$. In the language of quivers this would be represented by two vertices and $n - 2$ arrows from one to the other. With respect to multiplication from the left, we have $\mathrm{Tr}(\rho(e_1)) = n - 1$, $\mathrm{Tr}(\rho(e_2)) = 1$ and $\mathrm{Tr}(\rho(v)) = 0$ for all $v \in J$. Thus the trace zero module E is generated by J and by $w_1 = e_1 - (n-1)e_2$. The kernel of multiplication of J by E from the left is J, and for any element w of E, we have $w \equiv tw_1 \pmod J$ if and only if $w \cdot v = tv$ for any $v \in J$. Conversely, if $E = \tilde{V}_0$, then any non-zero element s of E^* determines a multiplication structure on $\boldsymbol{E} = V_0$ by setting $J = \ker(s)$, and defining

$$v \cdot v' = 0 \qquad \text{for } v, v' \in J,$$

$$w \cdot v = s(w)v \qquad \text{for } v \in J, \quad w \in E,$$

$$v \cdot w = -s(w)(n-1)v \qquad \text{for } v \in J, \quad w \in E,$$

$$w \cdot w = -s(w)(n-2)w + s(w)^2(n-1)e_0, \qquad \text{for } w \notin J.$$

The formula for w^2 is a direct result of the orthogonality of the two semi-idempotents

$$s(w)ne_1 = w + s(w)(n-1)e_0,$$

$$-s(w)ne_2 = w - s(w)e_0.$$

When $s = 0$, then the multiplication reduces to that of the n-fold point. The Kronecker algebra has no deformations and no specializations except the n-fold point, so the Kronecker component of C^0_n is isomorphic to E^*. If we sheafify we find that the Kronecker component of $\mathscr{K}^0_{n-1}(\mathscr{E})$ is just $\mathscr{E}^*$, as in the $n = 3$ case.

In general, in all the orbits of C^0_n which we have calculated so far, we find (1) that resolving the singularities of the closure of the orbit produces a fiber bundle over the flag variety for the flag $\mathscr{C} \subset J \subset J^2 \cdots$, (2) that the structure of the general fiber in this bundle is a reflection of the structure of the general algebra in the orbit, and (3) that this bundle can be sheafified in a natural way.

§5. Geometric n-covers

We turn to the last topic in this work, a geometric treatment of global n-covers. We have worked until now in the framework of conventional modern algebraic geometry, in which the spectrum always refers to the prime spectrum. We would like now to switch to the context of geometry on affine rings as given in the work of Artin and Schelter [1] and to work with the maximal spectrum. Although this is denoted in the literature on non-commutative algebraic geometry by Spec, we will denote it by Max Spec to avoid inconsistencies of notation within this paper.

We give a geometric version of the definition of an n-cover, first in the affine and then in the global case. We recall that an affine K-algebra is one which is finitely generated as an algebra over K. In the commutative case it is said to be of finite type over K.

DEFINITION. Let $f: A \to B$ be an n-cover of a commutative affine integral domain A, as defined in §1. The corresponding morphism of maximal spectra

$$f: \operatorname{Max Spec}(B) \to \operatorname{Max Spec}(A)$$

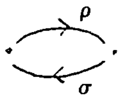
will be called an *affine n-cover.*

REMARK. Since the algebra homomorphism $f: A \to B$ maps A into the center of B, it is an extension ([1], p. 290), and thus the correspondence between the maximal spectra is actually a function.

DEFINITION. Let $X = \operatorname{Spec}(B)$, and $Y = \operatorname{Spec}(A)$. The X/Y-topology on X has as its open sets the sets $f^{-1}(U)$ for U open in Y.

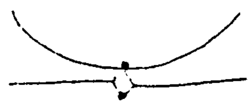
REMARK. Since B is a finitely generated module over A, the closed subsets in the X/Y topology are finite unions of Zariski closed subsets of $\operatorname{Spec}(B)$.

EXAMPLE 3. Let $A = K[t]$ be a polynomial ring in one variable, and let B be an algebra of rank 5 over A, whose general fiber is isomorphic to $M_2(K) \times K$, and whose special fiber at $t = 0$ is the five-dimensional algebra with two idempotents e_1, e_2 and two elements $\rho \in e_2 J e_1$ and $\sigma \in e_1 J e_2$ satisfying a relation $\rho\sigma = 0$. (This algebra is designated as A_4^* in [3].) For those familiar with representation theory, it has a quiver



and J^2 is generated by the product $\sigma\rho$.

Over the general point of MaxSpec(A·), MaxSpec(B) has two points, corresponding to $M_2(K)$ and K. At $t = 0$, the point corresponding to $M_2(K)$ splits into what Artin and Schelter call a "cluster of associated points", in this case corresponding to the two idempotents e_1 and e_2. Let W be the Zariski irreducible subset of MaxSpec(B) corresponding to the factor K. Then W intersects this cluster at the point corresponding to the idempotent e_1.



DEFINITION. Let Y be a commutative integeral scheme of finite type, and $\mathcal{F}$ a global n-cover. Let $\{U_i\}$ be an open cover of Y over which each $\mathcal{F}(U_i)$ is free. Let $X_i = \text{MaxSpec}(U_i)$ for each i. Each X_i is a sheaf of topological spaces with respect to two topologies, the Zariski topology and the X_i/U_i topology. The glueing algebra automorphisms φ_{ij} induce homomorphisms of topological spaces with respect to each of these topologies. Therefore the X_i glue together a topological space X with two topologies and a sheaf structure over Y. We can define a structure sheaf $\mathcal{O}_X(f^{-1}(U)) = \mathcal{F}(U)$ in the X/Y-topology.

EXAMPLE 4. Consider a global n-cover $\mathcal{F}$ which lies generically in the Kronecker component, and the corresponding space X. The generic fiber of $p: X \to Y$ has two points, corresponding to the two idempotents of the Kronecker algebra. The branch locus where these idempotents coalesce corresponds to the points where the algebra becomes local. This occurs when the section $\mathcal{E}$ corresponding to $f_1 - f_2$ goes to infinity, or, correspondingly, where the section $s \in \mathcal{E}^*$ defining the multiplication goes to zero.

Appendix 1

We wish to describe the procedure for constructing the parameter space to the versal deformation space, once the tangent space is known. The general procedure is given in Schlessinger's paper. We will make a slight adjustment in notation: the representing ring for C_n will be denoted simply by R (instead of R_n) and we set

$$R_2 = K[T_1, \dots, T_n]/(T_1, \dots, T_n)^2$$

where $T_1, \dots, T_N$ is a dual basis to the tangent space. We let α_2 be the tensor corresponding to the deformation over R_2 given by

$$x_i x_j = \sum \left(a_{ij}^k + \sum_{l=1}^{N} b_{ij}^{kl} T_l \right) x_k$$

where $\beta^l = (b_{ij}^{kl})$ is the structure constant tensor of a representative of the basis element corresponding to T_l. For any element v of the tangent space, substituting $T_l(v)$ for T_l, $l = 1, \dots, N$ will give a representative of v.

We set $S = K[[T_1, \dots, T_N]]$, $I = (T_1, \dots, T_N)$. Set $J_2 = I^2$, so that $R_2 = S/J_2$. We now proceed by induction. We want to define J_3 to be the largest ideal such that $J_2 \supset J_3 \supset I^3$, and α_2 lifts to a tensor α_3 over $R_3 = S/J_3$. In general, assume α_l and J_l have already been defined, we let J_{l+1} be the largest ideal such that $J_l \supset J_{l+1} \supset I^{l+1}$ and α_l lifts to α_{l+1} over $R_{l+1} = S/J_{l+1}$.

Schlessinger proves that such a J_l exists for each l. We then set $R_\infty = \lim R_l$ and let α_∞ be the corresponding limit of the α.

Since all the equations defining the representing ring are quadratic, we already have a great deal of information by the time we have constructed R_3, and we will describe explicitly how this is done.

In constructing the tangent space we took a general deformation tensor $\beta = (b_{ij}^k)$ for $i, j = 1, \dots, n-1$, $k = 0, \dots, n-1$. Letting the b_{ij}^k be variables, we generated two sets of equations: one, which we will call (*), consisted of homogeneous equations obtained from $\langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle = 0$. The second set, (**), was also homogeneous, derived from the automorphism relations. Since both sets were homogeneous, the solution space was a vector space, with certain of the parameters b_{ij}^k serving as free parameters $T_1, \dots, T_N$. Assuming that this calculation has already been carried out, we cease to think of the b_{ij}^k as variables and consider them linear functions of the T_i. In general each b_{ij}^p is either 0, $+T_l$ or $-T_l$, but occasionally we get more complicated linear combinations of the free parameters.

We now write $a_{ij}^p + b_{ij}^p + d_{ij}^p$ for the general element of the tensor α_3, where d_{ij}^p is a variable. We want to calculate the d_{ij}^p as quadratic functions of $T_1, \dots, T_N$. We no longer have to deal with eliminating automorphisms; that has already been done. Instead we get a non-homogeneous version (*)′ of the system (*):

$$\langle \alpha_3, \alpha_3 \rangle \equiv 0 \pmod{I^3}, \qquad \text{i.e.}$$

$$\langle \alpha + \beta + \delta, \alpha + \beta + \delta \rangle \equiv 0 \pmod{I^3}.$$

We already know that the order term $\langle \alpha, \alpha \rangle$ and the first order term $\langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ are both zero. We must discover when the second order term

$$\langle \beta, \beta \rangle + \langle \alpha, \delta \rangle + \langle \delta, \alpha \rangle \equiv 0 \pmod{I^3}.$$

We thus have the non-homogeneous system

$$\langle \alpha, \delta \rangle + \langle \delta, \alpha \rangle \equiv -\langle \beta, \beta \rangle.$$

J_3 is the largest ideal for which this system is consistent. We begin solving the right-hand side and whenever we uncover an equation $0 = \Sigma\, q_{ij} T_i T_j$ we add this quadratic expression in the free parameters to the ideal J_3, simplifying subsequent equations accordingly. We can make a further simplification. We need to find only one solution to the system. From any particular solution we can get many other solutions by adding solutions to the homogeneous system (*). The solution we found to (*) and (**) simultaneously is a solution to (*). Suppose b^p_{ij} was one of the variables we chose to be a free variable T_l.

We can create a solution to (*) by substituting $-\delta^p_{ij}$ for T_l and 0 for every other free parameter T_k. Thus, whenever we have a solution (δ) to $(*)'$ we will be able to replace it by a solution (δ') in which $\delta^p_{ij} = 0$. This can be done independently for each of $T_1, \ldots, T_l$. Thus we may simplify our system at the outset by assuming $\delta^p_{ij} = 0$ for those i, j, p for which b^p_{ij} was chosen as a free parameter. We could cut down the number of variables sill further if we would first find the general solution of (*) and only add the equations for (**) afterwards. However, this would increase the calculation time in the computation of the tangent space. With a little experimentation it should be possible to determine which way is better.

Appendix 2

In the case where the algebra being deformed has several idempotents, it is possible to reduce the number of equations to be solved. In a recent preprint [10] the author has shown that any basis which is partitioned into blocks by a Peirce decomposition can be deformed to a basis with the same property. Similarly it is shown that matrix units can be deformed to matrix units. The deformation of the matrix units allows us to do a sort of Morita reduction to a basic algebra. To construct the deformation of the basic algebra, we can restrict ourselves to deformations of the radical which preserve the Peirce decomposition. The only infinitesimal deformations by which we must divide are deformations $I + \varepsilon M$ when M preserves the Peirce decomposition. This provides a considerable reduction in the number of equations to be considered. With this reduction, deformations of algebras with 3 or 4 idempotents for dimensions up to around fifteen can probably be calculated with the currently operating program.

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